Decomposition of cuspidal space

Number Theory
Seminar

1 Apr 2021

Recall: k number field $\qquad G = GL_2$

$$\underbrace{L^2_0(G(k)\backslash G(\mathbb{A}), \omega)}_{\text{cuspidal functions}} \subset L^2(G(k)\backslash G(\mathbb{A}), \omega)$$

Action of $C_c(G(\mathbb{A})) \ni f$ on $L^2_0(G(k)\backslash G(\mathbb{A}), \omega)$

$$T_\omega(f)\varphi(x) = \int_{G(\mathbb{A})} f(y)\varphi(xy)\,dy$$

<u>Theorem</u>: For any $f \in C_c(G(\mathbb{A}))$, $T_\omega(f)$ is a compact operator on $L^2_0(G(k)\backslash G(\mathbb{A}), \omega)$.

<u>Corollary</u>: The unitary rep of $G(\mathbb{A})$ on $L^2_0(G(k)\backslash G(\mathbb{A}), \omega)$ decomposes into a Hilbert space ~~direct~~ discrete sum of top. irreducible reps, each occurring with finite multiplicity:

$$L^2_0(G(k)\backslash G(\mathbb{A}), \omega) = \widehat{\bigoplus_{n \in \text{countable set}}} \mathcal{H}_n \quad (\text{separability})$$

(See also Bump Theorem 3.3.2)

<u>Proof</u>: Define

$$\Sigma = \left\{ S = \text{a set of mutually orthogonal irred } G(\mathbb{A})\text{-submodules of } L^2_0(G(k)\backslash G(\mathbb{A}), \omega) \right\}$$

partially ordered by inclusion.

Zorn $\Rightarrow$ $\exists$ maximal element S.

S maximal

Let C be the orthogonal complement of $\overline{\bigoplus_{V \in S} V} \subset L^2_0(\ldots)$

Claim: $C = 0$ Suppose not,

Let $0 \neq \varphi \in C$. $\qquad f(g^{-1}) = f(g)\ \forall g \in G(\mathbb{A})$

Choose $f \in C_c(G(\mathbb{A}))$ such that $T_\omega(f)$ is self-adjoint and $T_\omega(f)\varphi \neq 0$. Write $T = T_\omega(f)$ for short.

Since $T|_C$ is compact and nonzero, by Spectral Theorem it has at least one nonzero eigenvalue λ.

Let $L \subset C$ be the λ-eigenspace; $\dim L < \infty$.

Let $L_0 \subset L$ be minimal w.r.t. $0 \neq L_0 = L \cap W$

$0 \neq W \subset C$, W $G(\mathbb{A})$-inv. subspace.

Let
$$\boxed{V = \bigcap_{\substack{W \subset C \\ W \cap L = L_0}} W}.$$

Claim: V is irreducible, $G(\mathbb{A})$-invariant.
This contradicts the maximality of S.

Suppose $\boxed{V = V_1 \oplus V_2}$ $\quad V_i \neq 0$ $G(\mathbb{A})$-invariant, invariant under T

Pick $0 \neq \varphi_0 \in L_0$ $\quad 0 \neq \varphi_0 = \varphi_1 + \varphi_2$ $\qquad \varphi_i \in V_i$

$T\varphi_i - \lambda\varphi_i \in V_i$, $i = 1,2$

$$\underbrace{(T\varphi_1 - \lambda\varphi_1)}_{\in V_1} + \underbrace{(T\varphi_2 - \lambda\varphi_2)}_{\in V_2} = T\varphi_0 - \lambda\varphi_0 = 0$$

$\Rightarrow T\varphi_1 = \lambda\varphi_1$ & $T\varphi_2 = \lambda\varphi_2$

WLOG $\varphi_1 \neq 0$ $\quad \varphi_1 \in L \cap V_1 = L_0$ $\qquad V_1 \subsetneq V$ contradicts $V = \bigcap_{W \cap L = L_0} W$

$\square$

② Local decomposition of irreps of $G(\mathbb{A})$
(Bump: Tensor product theorem, Thm 3.3.3)

Theorem: If ρ is an admissible irreducible unitary rep of $G(\mathbb{A})$, then ρ can be decomposed uniquely into

$$\rho = \hat{\bigotimes_v} \rho_v . \qquad \rho_v \text{ irrep of } G(k_v).$$

The setup: $\qquad G = GL_2$

For each place v, have

$G_v = G(k_v)$, $\quad K_v = G(\mathcal{O}_{k_v})$ for v finite

Given $(\rho_v, \mathcal{H}_v)$ irred. unitary rep of G_v with ρ_v admissible (multiplicity of any irrep θ_v of K_v in $\rho_v|_{K_v}$ is finite; write $\mathcal{H}_v(\theta_v)$ = isotypical comp in $\rho_v|_{K_v}$)

(Suppose $\mathcal{H}_v(\mathrm{id}) \neq 0$ for almost all v.)

For a.a. v, Fix unit vectors $\xi_v^\circ \in \mathcal{H}_v$ s.t. $\rho_v(k)\xi_v^\circ = \xi_v^\circ \quad \forall k \in K_v$

Define a rep $\rho = \hat{\bigotimes_v} \rho_v$ of $G(\mathbb{A})$ by

$$\hat{\bigotimes} \rho_v \, (g_v) \xi_v = \bigotimes_v \left(\rho_v(g_v) \xi_v \right)$$